

# On a Robin problem with $p$ -Laplacian and reaction bounded only from above

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## Abstract

The existence of three smooth solutions, one negative, one positive, and one nodal, to a homogeneous Robin problem with  $p$ -Laplacian and Carathéodory reaction is established. No sub-critical growth condition is taken on. Proofs exploit variational as well as truncation techniques. The case  $p = 2$  is separately examined, obtaining a further nodal solution via Morse's theory.

**Keywords:**  $p$ -Laplacian, Robin problem, constant-sign solution, nodal solution

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## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with a smooth boundary  $\partial\Omega$ , let  $1 < p < \infty$ , let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function, and let  $\beta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}_0^+)$  for some  $\alpha \in (0, 1)$ .

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Consider the homogeneous Robin problem

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Delta_p$  denotes the  $p$ -Laplace differential operator, namely  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$  for all  $u \in W^{1,p}(\Omega)$ , while  $\frac{\partial u}{\partial n_p} := |\nabla u|^{p-2}\nabla u \cdot n$ , with  $n(x)$  being the outward unit normal vector to  $\partial\Omega$  at its point  $x$ . As in [12, p. 1066],  $u \in W^{1,p}(\Omega)$  is called a (weak) solution of (1.1) provided

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta|u|^{p-2}uv \, d\sigma = \int_{\Omega} f(x, u)v \, dx \quad \forall v \in W^{1,p}(\Omega).$$

Equations driven by  $p$ -Laplacian type operators have been widely investigated under Dirichlet boundary conditions, mainly through variational, sub-super-solutions, and truncation techniques [18, 6, 7], besides Morse's theory [23]. There is a wealth of good results and the relevant literature looks daily increasing. On the other hand, these methods cannot always be adapted in a simple way to treat Neumann (i.e.,  $\beta \equiv 0$ ), or more generally Robin ( $\beta \not\equiv 0$ ), problems. That's why over the last few years the study of (1.1) has been receiving attention and very nice papers are already available. The more close to our work are [5, 11, 19] and, above all, [22]. Indeed, here, we prove the existence of three  $C^1$ -solutions to Problem (1.1), one positive, one negative, and one nodal, without assuming that  $t \mapsto f(x, t)$  exhibits a sub-critical behavior but is merely bounded on bounded sets. Moreover, roughly speaking, we suppose that

$$\limsup_{t \rightarrow \pm\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq a_0 < \lambda_1 \quad \text{and} \quad \lambda_2 < a_1 \leq \liminf_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} \leq \limsup_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} \leq a_2 < +\infty$$

uniformly in  $x \in \Omega$ , with  $\lambda_1$  (respectively,  $\lambda_2$ ) being the first (respectively, second) eigenvalue of  $(-\Delta_p, W^{1,p}(\Omega))$  under Robin's boundary condition; see Section 3 for precise formulations. So, no global growth from below is imposed on  $t \mapsto f(x, t)$ . The meaningful special case

$$f(x, t) := \lambda|t|^{p-2}t - g(x, t). \quad (x, t) \in \Omega \times \mathbb{R},$$

where  $\lambda > \lambda_2$ , is also examined and some results of [22] extended; cf. also [5, 11, 19], which however require  $\beta \equiv 0$ . When  $p = 2$  we obtain a second nodal solution by assuming, among other things,  $f(x, \cdot) \in C^1(\mathbb{R})$  and

$$|f'_t(x, t)| \leq a_3(1 + |t|^{r-2}) \quad \forall (x, t) \in \Omega \times \mathbb{R},$$

with  $2 \leq r < 2^*$ . Let us finally point out that an analogous investigation might be performed for the problem

$$-\Delta_p u + a(x)|u|^{p-2}u = f(x, u) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n_p} = 0 \quad \text{on } \partial\Omega,$$

where  $a \in L^\infty(\Omega)$  may change sign, exploiting the results of [20].

## 2 Preliminaries

Let  $(X, \|\cdot\|)$  be a real Banach space. Given a set  $V \subseteq X$ , write  $\overline{V}$  for the closure of  $V$ ,  $\partial V$  for the boundary of  $V$ , and  $\text{int}_X(V)$  or simply  $\text{int}(V)$ , when no confusion can arise, for the interior of  $V$ . If  $x \in X$  and  $\delta > 0$  then

$$B_\delta(x) := \{z \in X : \|z - x\| < \delta\}.$$

The symbol  $(X^*, \|\cdot\|_{X^*})$  denotes the dual space of  $X$ ,  $\langle \cdot, \cdot \rangle$  indicates the duality pairing between  $X$  and  $X^*$ , while  $x_n \rightarrow x$  (respectively,  $x_n \rightharpoonup x$ ) in  $X$  means ‘the sequence  $\{x_n\}$  converges strongly (respectively, weakly) in  $X$ ’.

Let  $T$  be a topological space and let  $L$  be a multifunction from  $T$  into  $X$  (briefly,  $L : T \rightarrow 2^X$ ), namely a function which assigns to each  $t \in T$  a nonempty subset  $L(t)$  of  $X$ . We say that  $L$  is lower semi-continuous when  $\{t \in T : L(t) \cap V \neq \emptyset\}$  turns out to be open in  $T$  for every open set  $V \subseteq X$ . A function  $l : T \rightarrow X$  is called a selection of  $L$  provided  $l(t) \in L(t)$  for all  $t \in T$ .

We say that  $\Phi : X \rightarrow \mathbb{R}$  is coercive iff

$$\lim_{\|x\| \rightarrow +\infty} \Phi(x) = +\infty,$$

while  $\Phi$  is called weakly sequentially lower semi-continuous when  $x_n \rightharpoonup x$  in  $X$  implies  $\Phi(x) \leq \liminf_{n \rightarrow \infty} \Phi(x_n)$ . Let  $\Phi \in C^1(X)$ . The classical Palais-Smale compactness condition for  $\Phi$  reads as follows.

(PS) *Every sequence  $\{x_n\} \subseteq X$  such that  $\{\Phi(x_n)\}$  is bounded and  $\|\Phi'(x_n)\|_{X^*} \rightarrow 0$  has a convergent subsequence.*

Define, provided  $c \in \mathbb{R}$ ,

$$\Phi^c := \{x \in X : \Phi(x) \leq c\}, \quad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),$$

where, as usual,  $K(\Phi)$  denotes the critical set of  $\Phi$ , i.e.,  $K(\Phi) := \{x \in X : \Phi'(x) = 0\}$ .

We say that  $A : X \rightarrow X^*$  is of type  $(S)_+$  iff

$$x_n \rightharpoonup x \text{ in } X, \quad \limsup_{n \rightarrow +\infty} \langle A(x_n), x_n - x \rangle \leq 0$$

imply  $x_n \rightarrow x$ . The next elementary result [15, Proposition 2.2] will be employed later.

*Proposition 2.1.* Let  $X$  be reflexive and let  $\Phi \in C^1(X)$  be coercive. Assume  $\Phi' = A + B$ , with  $A : X \rightarrow X^*$  of type  $(S)_+$  and  $B : X \rightarrow X^*$  compact. Then  $\Phi$  satisfies (PS).

Given a topological pair  $(A, B)$  fulfilling  $B \subset A \subseteq X$ , the symbol  $H_q(A, B)$ ,  $q \in \mathbb{N}_0$ , indicates the  $q^{\text{th}}$ -relative singular homology group of  $(A, B)$  with integer coefficients. If  $x_0 \in K_c(\Phi)$  is an isolated point of  $K(\Phi)$  then

$$C_q(\Phi, x_0) := H_q(\Phi^c \cap V, \Phi^c \cap V \setminus \{x_0\}), \quad q \in \mathbb{N}_0,$$

are the critical groups of  $\Phi$  at  $x_0$ . Here,  $V$  stands for any neighborhood of  $x_0$  such that  $K(\Phi) \cap \Phi^c \cap V = \{x_0\}$ . By excision, this definition does not depend on the choice of  $V$ . Suppose  $\Phi$  satisfies Condition (PS),  $\Phi|_{K(\Phi)}$  is bounded below, and  $c < \inf_{x \in K(\Phi)} \Phi(x)$ . Put

$$C_q(\Phi, \infty) := H_q(X, \Phi^c), \quad q \in \mathbb{N}_0.$$

The second deformation lemma [10, Theorem 5.1.33] implies that this definition does not depend on the choice of  $c$ . If  $K(\Phi)$  is finite, then setting

$$M(t, x) := \sum_{q=0}^{+\infty} \text{rank } C_q(\Phi, x) t^q, \quad P(t, \infty) := \sum_{q=0}^{+\infty} \text{rank } C_q(\Phi, \infty) t^q \quad \forall (t, x) \in \mathbb{R} \times K(\Phi),$$

the following Morse relation holds:

$$\sum_{x \in K(\Phi)} M(t, x) = P(t, \infty) + (1+t)Q(t), \quad (2.1)$$

where  $Q(t)$  denotes a formal series with nonnegative integer coefficients; see for instance [18, Theorem 6.62].

Now, let  $X$  be a Hilbert space, let  $x \in K(\Phi)$ , and let  $\Phi$  be  $C^2$  in a neighborhood of  $x$ . If  $\Phi''(x)$  turns out to be invertible, then  $x$  is called non-degenerate. The Morse index  $d$  of  $x$  is the supremum of the dimensions of the vector subspaces of  $X$  on which  $\Phi''(x)$  turns out to be negative definite. When  $x$  is non-degenerate and with Morse index  $d$  one has

$$C_q(\Phi, x) = \delta_{q,d} \mathbb{Z}, \quad q \in \mathbb{N}_0. \quad (2.2)$$

The monographs [16, 18] represent general references on the subject.

Throughout the paper,  $\Omega$  denotes a bounded domain of the real euclidean  $N$ -space  $(\mathbb{R}^N, |\cdot|)$  whose boundary is  $C^2$  while  $\beta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}_0^+)$  for some  $\alpha \in (0, 1)$  and  $\beta \not\equiv 0$ . On  $\partial\Omega$  we will employ the  $(N-1)$ -dimensional Hausdorff measure  $\sigma$ . The symbol  $m$  stands for the Lebesgue measure,  $p \in (1, +\infty)$ ,  $p' := p/(p-1)$ ,  $\|\cdot\|_q$  with  $q \geq 1$  indicates the usual norm of  $L^q(\Omega)$ ,  $X := W^{1,p}(\Omega)$ , and

$$\|u\| := (\|\nabla u\|_p^p + \|u\|_p^p)^{1/p}, \quad u \in X,$$

$$C_+ := \{u \in C^0(\overline{\Omega}) : u(x) \geq 0 \quad \forall x \in \overline{\Omega}\}.$$

Write  $p^*$  for the critical exponent of the Sobolev embedding  $W^{1,p}(\Omega) \subseteq L^q(\Omega)$ . Recall that  $p^* = Np/(N-p)$  if  $p < N$ ,  $p^* = +\infty$  otherwise, and the embedding is compact whenever  $1 \leq q < p^*$ . Moreover,

$$\text{int}(C_+) = \{u \in C_+ : u(x) > 0 \quad \forall x \in \overline{\Omega}\}.$$

Given  $t \in \mathbb{R}$ ,  $u, v : \Omega \rightarrow \mathbb{R}$ , and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ , define

$$t^\pm := \max\{\pm t, 0\}, \quad u^\pm(x) := u(x)^\pm, \quad N_f(u)(x) := f(x, u(x)).$$

The symbol  $u \leq v$  means  $u(x) \leq v(x)$  for almost every  $x \in \Omega$ . If  $u, v$  belong to a function space  $Y$  and  $u \leq v$  then we set

$$[u, v] := \{w \in Y : u \leq w \leq v\}.$$

Let  $A_p : X \rightarrow X^*$  be the nonlinear operator stemming from the negative  $p$ -Laplacian  $\Delta_p$ , i.e.,

$$\langle A_p(u), v \rangle := \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx \quad \forall u, v \in X.$$

A standard argument [18, Proposition 2.71] ensures that  $A_p$  is of type (S)<sub>+</sub>.

*Remark 2.1.* Given  $u \in X$  and  $w \in L^{p'}(\Omega)$ , the condition

$$\langle A_p(u), v \rangle + \int_{\partial\Omega} \beta(x) |u(x)|^{p-2} u(x) v(x) d\sigma = \int_{\Omega} w(x) v(x) dx, \quad v \in X,$$

is equivalent to

$$-\Delta_p u = w \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n_p} + \beta(x) |u|^{p-2} u = 0 \quad \text{on } \partial\Omega.$$

This easily comes out from the nonlinear Green's identity [10, Theorem 2.4.54]; see for instance the proof of [22, Proposition 3].

We shall employ some facts on the spectrum  $\sigma(-\Delta_p)$  of the operator  $-\Delta_p$  with homogeneous Robin boundary conditions. So, consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(x) |u|^{p-2} u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

The Liusternik-Schnirelman theory provides a strictly increasing sequence  $\{\lambda_n\} \subseteq \mathbb{R}_0^+$  of eigenvalues for (2.3). Denote by  $E(\lambda_n)$  the eigenspace corresponding to  $\lambda_n$ ,  $n \in \mathbb{N}$ . From [12, 22] we know that:

(p<sub>1</sub>)  $\lambda_1$  is positive, isolated, and simple. Further,

$$\lambda_1 = \inf \left\{ \frac{\|\nabla u\|_p^p + \int_{\partial\Omega} \beta |u|^p d\sigma}{\|u\|_p^p} : u \in X, u \neq 0 \right\}.$$

(p<sub>2</sub>) There exists an  $L^p$ -normalized eigenfunction  $\hat{u}_1 \in \text{int}(C_+)$  associated with  $\lambda_1$ .

The next characterization of  $\lambda_2$  will be used later. For its proof we refer the reader to [22, Proposition 5].

(p<sub>3</sub>) Write  $U := \{u \in X : \|u\|_p = 1\}$  as well as

$$\Gamma_1 := \{\gamma \in C^0([-1, 1], U) : \gamma(-1) = -\hat{u}_1, \gamma(1) = \hat{u}_1\},$$

$$\Phi(u) := \|\nabla u\|_p^p + \int_{\partial\Omega} \beta(x) |u(x)|^p d\sigma, \quad u \in X.$$

Then

$$\lambda_2 = \inf_{\gamma \in \Gamma_1} \max_{t \in [-1, 1]} \Phi(\gamma(t)).$$

Define  $U_C := \{u \in C^1(\overline{\Omega}) : \|u\|_p = 1\}$ . Evidently,  $U_C$  turns out to be dense in  $U$ . Let

$$\Gamma_C := \{\gamma \in C^0([-1, 1], U_C) : \gamma(-1) = -\hat{u}_1, \gamma(1) = \hat{u}_1\}$$

**Lemma 2.1.** *The set  $\Gamma_C$  is dense in  $\Gamma_1$  with respect to the usual norm of  $C^0([-1, 1], X)$ .*

*Proof.* Pick any  $\gamma \in \Gamma_1$ . We shall prove that there exists a sequence  $\{\gamma_n\} \subseteq \Gamma_C$  fulfilling

$$\lim_{n \rightarrow +\infty} \max_{t \in [-1, 1]} \|\gamma_n(t) - \gamma(t)\| = 0. \quad (2.4)$$

The multifunction  $L_n : [-1, 1] \rightarrow 2^{C^1(\overline{\Omega})}$  defined by

$$L_n(t) := \begin{cases} \{-\hat{u}_1\} & \text{when } t = -1, \\ \{u \in C^1(\overline{\Omega}) : \|u - \gamma(t)\| < 1/n\} & \text{if } t \in (-1, 1), \\ \{\hat{u}_1\} & \text{when } t = 1 \end{cases}$$

takes nonempty convex values and is lower semi-continuous. So, Theorem 3.1''' in [17] provides a continuous selection  $l_n : [-1, 1] \rightarrow C^1(\overline{\Omega})$  of  $L_n$ . This entails

$$\|l_n(t) - \gamma(t)\| < \frac{1}{n} \quad \forall t \in (-1, 1), \quad l_n(-1) = -\hat{u}_1, \quad l_n(1) = \hat{u}_1. \quad (2.5)$$

Consequently,

$$\lim_{n \rightarrow +\infty} \|l_n(t)\|_p = \|\gamma(t)\|_p = 1 \quad (2.6)$$

uniformly with respect to  $t \in [-1, 1]$ . For any  $n$  large enough we can thus set

$$\gamma_n(t) := \frac{l_n(t)}{\|l_n(t)\|_p}, \quad t \in [-1, 1].$$

On account of (2.5) and (p<sub>3</sub>) one has  $\gamma_n \in \Gamma_C$ . Moreover,

$$\begin{aligned} \|\gamma_n(t) - \gamma(t)\| &\leq \|\gamma_n(t) - l_n(t)\| + \|l_n(t) - \gamma(t)\| \\ &< |1 - \|l_n(t)\|_p| \frac{\|l_n(t)\|}{\|l_n(t)\|_p} + \frac{1}{n} \quad \forall t \in [-1, 1]. \end{aligned} \quad (2.7)$$

Recall that  $\gamma \in \Gamma_1$ . Since, by (2.5) again,

$$\begin{aligned} \max_{t \in [-1, 1]} |1 - \|l_n(t)\|_p| &= \max_{t \in [-1, 1]} \left| \|\gamma(t)\|_p - \|l_n(t)\|_p \right| \\ &\leq \max_{t \in [-1, 1]} \|\gamma(t) - l_n(t)\|_p \leq c \max_{t \in [-1, 1]} \|\gamma(t) - l_n(t)\| \leq \frac{c}{n} \end{aligned}$$

for some  $c > 0$ , (2.4) immediately follows from (2.5)–(2.7).  $\square$

Finally, it is known [12, Section 4] that

$$(p_4) \quad E(\lambda_n) \subseteq C^1(\overline{\Omega}) \text{ for all } n \in \mathbb{N}.$$

Let  $p := 2$ . Through [9, Proposition 3] we also obtain

$$(p_5) \quad \text{If } u \text{ lies in } E(\lambda_n) \text{ and vanishes on a set of positive Lebesgue measure then } u = 0.$$

Setting

$$\bar{H}_n := \oplus_{m=1}^n E(\lambda_m), \quad \hat{H}_n := \bar{H}_n^\perp,$$

each  $u \in H^1(\Omega)$  can uniquely be written as  $u = \bar{u} + \hat{u}$ , with  $\bar{u} \in \bar{H}_n$  and  $\hat{u} \in \hat{H}_n$ , because  $H^1(\Omega) = \bar{H}_n \oplus \hat{H}_n$ . By orthogonality one has, for every  $n \geq 2$ ,

$$\begin{aligned} \lambda_n &= \max \left\{ \frac{\|\nabla \bar{u}\|_2^2 + \int_{\partial\Omega} \beta \bar{u}^2 d\sigma}{\|\bar{u}\|_2^2} : \bar{u} \in \bar{H}_n, \bar{u} \neq 0 \right\} \\ &= \min \left\{ \frac{\|\nabla \hat{u}\|_2^2 + \int_{\partial\Omega} \beta \hat{u}^2 d\sigma}{\|\hat{u}\|_2^2} : \hat{u} \in \hat{H}_{n-1}, \hat{u} \neq 0 \right\}. \end{aligned} \tag{2.8}$$

A simple argument, based on orthogonality and (p<sub>5</sub>), yields the next result.

**Lemma 2.2.** *Let  $n \in \mathbb{N}$  and let  $\theta \in L^\infty(\Omega) \setminus \{\lambda_n\}$  satisfy  $\theta \geq \lambda_n$ . Then there exists a constant  $\bar{c} > 0$  such that*

$$\|\nabla \bar{u}\|_2^2 + \int_{\partial\Omega} \beta(x) \bar{u}(x)^2 d\sigma - \int_{\Omega} \theta(x) \bar{u}(x)^2 dx \leq -\bar{c} \|\bar{u}\|^2 \quad \forall \bar{u} \in \bar{H}_n.$$

*Let  $n \in \mathbb{N}$  and let  $\theta \in L^\infty(\Omega) \setminus \{\lambda_{n+1}\}$  satisfy  $\theta \leq \lambda_{n+1}$ . Then there exists a constant  $\hat{c} > 0$  such that*

$$\|\nabla \hat{u}\|_2^2 + \int_{\partial\Omega} \beta(x) \hat{u}(x)^2 d\sigma - \int_{\Omega} \theta(x) \hat{u}(x)^2 dx \geq \hat{c} \|\hat{u}\|^2 \quad \forall \hat{u} \in \hat{H}_n.$$

### 3 Existence results

To avoid unnecessary technicalities, ‘for every  $x \in \Omega$ ’ will take the place of ‘for almost every  $x \in \Omega$ ’ and the variable  $x$  will be omitted when no confusion can arise.

Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that  $f(\cdot, 0) = 0$  and let

$$F(x, \xi) := \int_0^\xi f(x, t) dt, \quad (x, \xi) \in \Omega \times \mathbb{R}. \tag{3.1}$$

We will posit the following assumptions.

(f<sub>1</sub>) *To every  $\rho > 0$  there corresponds  $a_\rho \in L^\infty(\Omega)$  satisfying  $\sup_{|t| \leq \rho} |f(x, t)| \leq a_\rho(x)$  in  $\Omega$ .*

(f<sub>2</sub>)  *$\limsup_{t \rightarrow \pm\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq a_0 < \lambda_1$  uniformly with respect to  $x \in \Omega$ .*



(f<sub>3</sub>) There exist  $a_1, a_2 \in L^\infty(\Omega) \setminus \{\lambda_1\}$  such that  $\lambda_1 \leq a_1 \leq a_2$  and

$$a_1(x) \leq \liminf_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} \leq \limsup_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} \leq a_2(x) \quad \text{uniformly in } x \in \Omega.$$

(f<sub>4</sub>) To every  $\rho > 0$  there corresponds  $\mu_\rho > 0$  such that  $t \mapsto f(x, t) + \mu_\rho |t|^{p-2}t$  is nondecreasing on  $[-\rho, \rho]$  for all  $x \in \Omega$ .

*Remark 3.1.* Obviously, both (f<sub>3</sub>) and (f<sub>4</sub>) imply that for each  $\rho > 0$  we can find  $\mu_\rho > 0$  fulfilling

$$f(x, t)t + \mu_\rho |t|^p \geq 0 \quad \forall (x, t) \in \Omega \times [-\rho, \rho].$$

Now, recall that  $X := W^{1,p}(\Omega)$ . The energy functional  $\varphi : X \rightarrow \mathbb{R}$  stemming from Problem (1.1) is

$$\varphi(u) := \frac{1}{p} \left( \|\nabla u\|_p^p + \int_{\partial\Omega} \beta(x) |u(x)|^p d\sigma \right) - \int_{\Omega} F(x, u(x)) dx, \quad u \in X, \quad (3.2)$$

with  $F$  given by (3.1). One clearly has  $\varphi \in C^1(X)$ . Moreover, if (f<sub>2</sub>) holds then, fixed any  $\hat{a}_0 \in (a_0, \lambda_1)$ , there exists  $M > 0$  such that

$$\frac{f(x, t)}{|t|^{p-2}t} < \hat{a}_0 < \lambda_1 \quad (3.3)$$

provided  $x \in \Omega$  and  $|t| \geq M$ . Since (p<sub>2</sub>) entails  $t_1 \hat{u}_1 \geq M$  for  $t_1 > 0$  large enough, inequality (3.3) combined with Remark 2.1 lead to

$$\int_{\Omega} f(x, \hat{u})v dx \leq \lambda_1 \int_{\Omega} \hat{u}^{p-1} dx = \langle A_p(\hat{u}), v \rangle + \int_{\partial\Omega} \beta \hat{u}^{p-1} v dx, \quad v \in X, v \geq 0, \quad (3.4)$$

where  $\hat{u} := t_1 \hat{u}_1$ .

### 3.1 Constant-sign solutions

Define, provided  $x \in \Omega$  and  $t, \xi \in \mathbb{R}$ ,

$$\hat{g}_+(x, t) := \begin{cases} 0 & \text{when } t < 0, \\ f(x, t) + t^{p-1} & \text{if } 0 \leq t \leq \hat{u}(x), \\ f(x, \hat{t}) + \hat{t}^{p-1} & \text{otherwise,} \end{cases} \quad (3.5)$$

$$\hat{g}_-(x, t) := \begin{cases} f(x, -\hat{t}) - \hat{t}^{p-1} & \text{when } t < -\hat{u}(x), \\ f(x, t) + |t|^{p-2}t & \text{if } -\hat{u}(x) \leq t \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

as well as

$$\hat{G}_{\pm}(x, \xi) := \int_0^{\xi} \hat{g}_{\pm}(x, t) dt.$$

It is evident that the corresponding truncated functionals

$$\hat{\psi}_{\pm}(u) := \frac{1}{p} \left( \|u\|^p + \int_{\partial\Omega} \beta(x) |u(x)|^p d\sigma \right) - \int_{\Omega} \hat{G}_{\pm}(x, u(x)) dx, \quad u \in X,$$

belong to  $C^1(X)$  also.

**Theorem 3.1.** *Under hypotheses  $(f_1)$ – $(f_3)$ , Problem (1.1) admits at least two constant-sign solutions  $u_0 \in [0, \hat{u}] \cap \text{int}(C_+)$  and  $v_0 \in [-\hat{u}, 0] \cap (-\text{int}(C_+))$ . If, moreover,  $(f_4)$  holds then  $u_0, v_0$  are local minimizers for  $\varphi$ .*

*Proof.* The space  $X$  compactly embeds in  $L^p(\Omega)$  while the Nemitskii operator  $N_{\hat{g}_+}$  turns out to be continuous on  $L^p(\Omega)$ . Thus, a standard argument ensures that  $\hat{\psi}_+$  is weakly sequentially lower semi-continuous. Since, on account of (3.5), it is coercive, we have

$$\inf_{u \in X} \hat{\psi}_+(u) = \hat{\psi}_+(u_0) \quad (3.6)$$

for some  $u_0 \in X$ . Fix  $\varepsilon > 0$ . Assumption  $(f_3)$  yields  $\delta > 0$  small such that

$$F(x, \xi) \geq \frac{a_1(x) - \varepsilon}{p} |\xi|^p \quad \forall (x, \xi) \in \Omega \times [-\delta, \delta].$$

If  $\tau \in (0, t_1)$  complies with  $\tau \hat{u}_1 \leq \delta$  then, by (3.5), the choice of  $\tau$ , the above inequality, and Remark 2.1,

$$\begin{aligned} \hat{\psi}_+(\tau \hat{u}_1) &\leq \frac{\tau^p}{p} \left( \|\nabla \hat{u}_1\|_p^p + \int_{\partial\Omega} \beta \hat{u}_1^p d\sigma - \int_{\Omega} (a_1 - \varepsilon) \hat{u}_1^p dx \right) \\ &= \frac{\tau^p}{p} \left( \lambda_1 \int_{\Omega} \hat{u}_1^p dx - \int_{\Omega} (a_1 - \varepsilon) \hat{u}_1^p dx \right) = \frac{\tau^p}{p} \left( \int_{\Omega} (\lambda_1 - a_1) \hat{u}_1^p dx + \varepsilon \right) < 0 \end{aligned}$$

as soon as  $\varepsilon < \int_{\Omega} (\lambda_1 - a_1) \hat{u}_1^p dx$ . Hence,

$$\hat{\psi}_+(u_0) < 0 = \hat{\psi}_+(0),$$

which clearly means  $u_0 \neq 0$ . Now, through (3.6) we get  $\hat{\psi}'_+(u_0) = 0$ , namely

$$\langle A_p(u_0) + |u_0|^{p-2} u_0, v \rangle + \int_{\partial\Omega} \beta |u_0|^{p-2} u_0 v d\sigma = \langle N_{\hat{g}_+}(u_0), v \rangle, \quad v \in X. \quad (3.7)$$

Choosing  $v := -u_0^-$  in (3.7) leads to  $\|\nabla u_0^-\|_p^p + \|u_0^-\|_p^p \leq 0$ , and  $u_0 \geq 0$ . Next, pick  $v := (u_0 - \hat{u})^+$ . From (3.7), besides (3.4), it follows

$$\begin{aligned} \langle A_p(u_0), (u_0 - \hat{u})^+ \rangle + \int_{\Omega} u_0^{p-1} (u_0 - \hat{u})^+ dx + \int_{\partial\Omega} \beta u_0^{p-1} (u_0 - \hat{u})^+ d\sigma \\ = \int_{\Omega} (f(x, \hat{u}) + \hat{u}^{p-1}) (u_0 - \hat{u})^+ dx \\ \leq \langle A_p(\hat{u}), (u_0 - \hat{u})^+ \rangle + \int_{\partial\Omega} \beta \hat{u}^{p-1} (u_0 - \hat{u})^+ d\sigma + \int_{\Omega} \hat{u}^{p-1} (u_0 - \hat{u})^+ dx, \end{aligned}$$

that is

$$\langle A_p(u_0) - A_p(\hat{u}), (u_0 - \hat{u})^+ \rangle + \int_{\partial\Omega} \beta (u_0^{p-1} - \hat{u}^{p-1}) (u_0 - \hat{u})^+ d\sigma + \int_{\Omega} (u_0^{p-1} - \hat{u}^{p-1}) (u_0 - \hat{u})^+ dx \leq 0.$$

Consequently,  $u_0 \leq \hat{u}$ . Now, (3.7) becomes

$$\langle A_p(u_0), v \rangle + \int_{\partial\Omega} \beta u_0^{p-1} v d\sigma = \int_{\Omega} f(x, u_0) v dx \quad \forall v \in X$$

whence, on account of Remark 2.1,

$$-\Delta_p u_0 = f(x, u_0) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n_p} + \beta(x) |u_0|^{p-2} u_0 = 0 \quad \text{on } \partial\Omega.$$

Standard regularity arguments ensure that  $u_0 \in C_+ \setminus \{0\}$ . Let  $\rho := \|\hat{u}\|_{\infty} \geq \|u_0\|_{\infty}$ . Due to Remark 3.1 one has

$$-\Delta_p u_0(x) + \mu_{\rho} u_0(x)^{p-1} = f(x, u_0(x)) + \mu_{\rho} u_0(x)^{p-1} \geq 0 \quad \text{a.e. in } \Omega.$$

Therefore, by [25, Theorem 5],  $u_0 \in \text{int}(C_+)$  and thus  $u_0 \in [0, \hat{u}] \cap \text{int}(C_+)$ , as desired. Define  $u_{\delta} := u_0 + \delta$ , where  $\delta > 0$ . Since

$$-\Delta_p u_{\delta}(x) + \mu_{\rho} u_{\delta}(x)^{p-1} \leq -\Delta_p u_0(x) + \mu_{\rho} u_0(x)^{p-1} + o(\delta) = f(x, u_0(x)) + \mu_{\rho} u_0(x)^{p-1} + o(\delta),$$

exploiting (f<sub>4</sub>) and (3.3) we obtain

$$\begin{aligned} -\Delta_p u_{\delta}(x) + \mu_{\rho} u_{\delta}(x)^{p-1} &\leq f(x, \hat{u}(x)) + \mu_{\rho} \hat{u}(x)^{p-1} + o(\delta) \\ &< (\hat{a}_0 + \mu_{\rho}) \hat{u}(x)^{p-1} + o(\delta) \leq (\lambda_1 + \mu_{\rho}) \hat{u}(x)^{p-1} = -\Delta_p \hat{u}(x) + \mu_{\rho} \hat{u}(x)^{p-1} \end{aligned}$$

for any  $\delta > 0$  small enough, because

$$(\lambda_1 - \hat{a}_0) \inf_{x \in \Omega} \hat{u}(x)^{p-1} > 0;$$

cf. (3.3) as well as (p<sub>2</sub>). Theorem 5 of [25] gives  $u_\delta \leq \hat{u}$ , whence

$$u_0 \in \text{int}_{C^1(\overline{\Omega})}([0, \hat{u}]). \quad (3.8)$$

Observe next that  $\varphi|_{[0, \hat{u}]} = \hat{\psi}_+|_{[0, \hat{u}]}$  thanks to (3.5). So, by (3.8) and (3.6), the function  $u_0$  is a  $C^1(\overline{\Omega})$ -local minimizer for  $\varphi$ . Finally, [22, Proposition 3] guarantees that the same holds putting  $X$  in place of  $C^1(\overline{\Omega})$ .

A similar argument produces  $v_0 \in [-\hat{u}, 0] \cap (-\text{int}(C_+))$  with the asserted properties.  $\square$

*Remark 3.2.* The upper bound at zero requested by (f<sub>3</sub>) for  $t \mapsto f(x, t)/|t|^{p-2}t$  has not been used to find constant-sign solutions.

The next result looks like [13, Theorem 3.3]; see also [22, Proposition 8]. So, we will only sketch its proof.

**Theorem 3.2.** *Let (f<sub>1</sub>)–(f<sub>3</sub>) be satisfied. Then (1.1) possesses the smallest (resp., the biggest) nontrivial solution  $u_*$  in  $[0, \hat{u}]$  (resp.,  $v_*$  in  $[-\hat{u}, 0]$ ). Further,  $-v_*, u_* \in \text{int}(C_+)$ .*

*Proof.* Define  $\Sigma_+ := \{u \in X \setminus \{0\} : u \text{ solves (1.1) and } 0 \leq u \leq \hat{u}\}$ . Due to Theorem 3.1 one has  $\Sigma_+ \neq \emptyset$ . Actually,  $\Sigma_+ \subseteq \text{int}(C_+)$ . The same arguments employed in establishing [2, Proposition 8] show here that

1)  $\Sigma_+$  is downward directed, and

2)  $\inf \Sigma_+ = \inf_{n \in \mathbb{N}} u_n = u_*$  for some  $\{u_n\} \subseteq \Sigma_+$ ,  $u_* \in X$  fulfilling  $u_n \rightarrow u_*$  in  $X$  and  $u_n(x) \rightarrow u_*(x)$  a.e. in  $\Omega$ .

Hence,  $u_*$  turns out to be a solution of (1.1) lying in  $[0, \hat{u}]$ . It remains to verify that  $u_* \neq 0$ . Suppose on the contrary  $u_* = 0$ . Reasoning exactly as in the proof of [1, Proposition 14] we obtain  $\alpha \in L^\infty(\Omega)$  and  $w \in \text{int}(C_+)$  with the properties below.

3)  $a_1 \leq \alpha \leq a_2$ .

4)  $-\Delta_p w(x) = \alpha(x)|w(x)|^{p-2}w(x)$  a.e. in  $\Omega$ ,  $\frac{\partial w}{\partial n_p} + \beta(x)|w|^{p-2}w = 0$  on  $\partial\Omega$ .

Let  $v \in \text{int}(C_+)$ . Gathering [3, Theorem 1.1] and [10, Theorem 2.4.54] together produce

$$\begin{aligned} 0 &\leq \int_{\Omega} \left( |\nabla v|^p - \nabla \left( \frac{v^p}{w^{p-1}} \right) \cdot |\nabla w|^{p-2} \nabla w \right) dx \\ &= \int_{\Omega} \left( |\nabla v|^p - \frac{v^p}{w^{p-1}} (-\Delta_p w) \right) dx - \int_{\partial\Omega} \frac{\partial u}{\partial n_p} \frac{v^p}{w^{p-1}} d\sigma \\ &= \int_{\Omega} (|\nabla v|^p - \alpha v^p) dx + \int_{\partial\Omega} \beta v^p d\sigma, \end{aligned}$$

where 4) has been used. If  $v = \hat{u}_1$  then, by 3) and (f<sub>3</sub>),

$$0 \leq \int_{\Omega} (\lambda_1 - \alpha(x)) \hat{u}_1(x)^p dx \leq \int_{\Omega} (\lambda_1 - a_1(x)) \hat{u}_1(x)^p dx < 0,$$

which is impossible. Therefore,  $u_* \in \Sigma_+$ , and the conclusion follows. A similar argument applies to get  $v_*$ .  $\square$

### 3.2 Nodal solutions

Define, for every  $x \in \Omega$  and  $t, \xi \in \mathbb{R}$ ,

$$\hat{f}(x, t) := \begin{cases} f(x, v_*(x)) + |v_*(x)|^{p-2} v_*(x) & \text{when } t < v_*(x), \\ f(x, t) + |t|^{p-2} t & \text{if } v_*(x) \leq t \leq u_*(x), \\ f(x, u_*(x)) + u_*(x)^{p-1} & \text{when } t > u_*(x), \end{cases} \quad (3.9)$$

$$\hat{f}_{\pm}(x, t) := \hat{f}(x, t^{\pm}),$$

as well as

$$\hat{F}(x, \xi) := \int_0^{\xi} \hat{f}(x, t) dt, \quad \hat{F}_{\pm}(x, \xi) := \int_0^{\xi} \hat{f}_{\pm}(x, t) dt.$$

It is evident that the corresponding truncated functionals

$$\hat{\varphi}(u) := \frac{1}{p} \left( \|u\|^p + \int_{\partial\Omega} \beta(x) |u(x)|^{p-1} d\sigma \right) - \int_{\Omega} \hat{F}(x, u(x)) dx, \quad u \in X,$$

$$\hat{\varphi}_{\pm}(u) := \frac{1}{p} \left( \|u\|^p + \int_{\partial\Omega} \beta(x) |u(x)|^{p-1} d\sigma \right) - \int_{\Omega} \hat{F}_{\pm}(x, u(x)) dx, \quad u \in X,$$

belong to  $C^1(X)$ .

**Lemma 3.1.** *Under hypotheses (f<sub>1</sub>)–(f<sub>3</sub>) one has*

$$K(\hat{\varphi}) \subseteq [v_*, u_*], \quad K(\hat{\varphi}_-) = \{0, v_*\}, \quad K(\hat{\varphi}_+) = \{0, u_*\}.$$

*Proof.* If  $u \in K(\hat{\varphi})$  then

$$\langle A_p(u) + |u|^{p-2} u, v \rangle + \int_{\partial\Omega} \beta |u|^{p-2} u v d\sigma = \langle N_{\hat{f}}(u), v \rangle \quad \forall v \in X.$$

Letting  $v := (u - u_*)^+$  yields

$$\begin{aligned} \langle A_p(u), (u - u_*)^+ \rangle + \int_{\Omega} u^{p-1} (u - u_*)^+ dx + \int_{\partial\Omega} \beta |u|^{p-1} (u - u_*)^+ d\sigma \\ = \int_{\Omega} (f(x, u_*) + u_*^{p-1}) (u - u_*)^+ dx. \end{aligned}$$

Since, by Theorem 3.2, the function  $u_*$  solves (1.1), this results in

$$\langle A_p(u) - A_p(u_*), (u - u_*)^+ \rangle + \int_{\Omega} (u^{p-1} - u_*^{p-1})(u - u_*)^+ dx + \int_{\partial\Omega} \beta(u^{p-1} - u_*^{p-1})(u - u_*)^+ d\sigma = 0.$$

Therefore,  $m(\{x \in \Omega : u(x) > u_*(x)\}) = 0$ , whence  $u \leq u_*$ . An analogous reasoning provides  $u \geq v_*$ , and the first inclusion holds.

As before, we obtain  $K(\hat{\varphi}_-) \subseteq [v_*, 0]$ , while the extremality of  $v_*$  (see Theorem 3.2) forces  $K(\hat{\varphi}_-) = \{v_*, 0\}$ . The remaining proof is similar.  $\square$

**Lemma 3.2.** *Let  $(f_1)-(f_4)$  be satisfied. Then  $u_*, v_*$  are local minimizers for  $\hat{\varphi}$ .*

*Proof.* The space  $X$  compactly embeds in  $L^p(\Omega)$  while the Nemitskii operator  $N_{\hat{f}_+}$  turns out to be continuous on  $L^p(\Omega)$ . Thus, a standard argument ensures that  $\hat{\varphi}_+$  is weakly sequentially lower semi-continuous. Since, on account of (3.9), it is coercive, we have

$$\inf_{u \in X} \hat{\varphi}_+(u) = \hat{\varphi}_+(u_0) \quad (3.10)$$

for some  $u_0 \in X$ . Reasoning as in the proof of Theorem 3.1 produces  $\hat{\varphi}_+(u_0) < 0$ , i.e.,  $u_0 \neq 0$ . Hence, by (3.10) and Lemma 3.1,  $u_0 = u_* \in \text{int}(C_+)$ . Since  $\hat{\varphi}|_{C_+} = \hat{\varphi}_+|_{C_+}$ , the function  $u_0$  turns out to be a  $C^1(\overline{\Omega})$ -local minimizer for  $\hat{\varphi}$ . Now, Proposition 3 in [22] guarantees that the same holds true with  $X$  in place of  $C^1(\overline{\Omega})$ . A similar argument applies to  $v_*$ .  $\square$

**Theorem 3.3.** *Under hypotheses  $(f_1)-(f_4)$ , with  $\text{ess inf}_{x \in \Omega} a_1(x) > \lambda_2$ , Problem (1.1) possesses a nodal solution  $u_1 \in [v_*, u_*] \cap C^1(\overline{\Omega})$ .*

*Proof.* By Theorem 3.2 and Lemma 3.1, we may assume  $K(\hat{\varphi})$  finite. Let  $\hat{\varphi}(v_*) \leq \hat{\varphi}(u_*)$  (the other case is analogous). Without loss of generality, the local minimizer  $u_*$  for  $\hat{\varphi}$  (cf. Lemma 3.2) can be supposed proper. Thus, there exists  $\rho \in (0, \|u_* - v_*\|)$  such that

$$\hat{\varphi}(u_*) < c_\rho := \inf_{u \in \partial B_\rho(u_*)} \hat{\varphi}(u). \quad (3.11)$$

Moreover,  $\hat{\varphi}$  fulfils Condition (PS) because, due to (3.9), it is coercive; see Proposition 2.1. So, the Mountain Pass Theorem yields a point  $u_1 \in X$  complying with  $\hat{\varphi}'(u_1) = 0$  and

$$c_\rho \leq \hat{\varphi}(u_1) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \hat{\varphi}(\gamma(t)), \quad (3.12)$$

where

$$\Gamma := \{\gamma \in C^0([0, 1], X) : \gamma(0) = v_*, \gamma(1) = u_*\}.$$

Obviously,  $u_1$  solves (1.1). Through (3.11)–(3.12), besides Lemma 3.1, we get

$$u_1 \in [v_*, u_*] \setminus \{v_*, u_*\},$$

while standard regularity arguments yield  $u_1 \in C^1(\overline{\Omega})$ . The proof is thus completed once one verifies that  $u_1 \neq 0$ . This will follow from the inequality

$$\hat{\varphi}(u_1) < 0, \quad (3.13)$$

which, in view of (3.12), can be shown by constructing a path  $\tilde{\gamma} \in \Gamma$  such that

$$\hat{\varphi}(\tilde{\gamma}(t)) < 0 \quad \forall t \in [0, 1]. \quad (3.14)$$

By (f<sub>3</sub>) to every  $\eta > 0$  there corresponds  $\delta > 0$  such that

$$F(x, \xi) \geq \frac{a_1(x) - \eta}{p} |\xi|^p, \quad (x, z) \in \Omega \times [-\delta, \delta]. \quad (3.15)$$

Combining (p<sub>4</sub>) with Lemma 2.1 entails

$$\max_{t \in [-1, 1]} \Phi(\gamma_\eta(t)) < \lambda_2 + \eta \quad (3.16)$$

for appropriate  $\gamma_\eta \in \Gamma_C$ . Since  $\gamma_\eta([-1, 1])$  is compact in  $C^1(\overline{\Omega})$  and  $-v_*, u_* \in \text{int}(C_+)$  we can find  $\varepsilon > 0$  so small that

$$v_*(x) \leq \varepsilon \gamma_\eta(t)(x) \leq u_*(x), \quad |\varepsilon \gamma_\eta(t)(x)| \leq \delta$$

whenever  $x \in \Omega$ ,  $t \in [-1, 1]$ . Thanks to (3.15)–(3.16) one has

$$\begin{aligned} \hat{\varphi}(\varepsilon \gamma_\eta(t)) &= \frac{\varepsilon^p}{p} \left( \|\gamma_\eta(t)\|^p + \int_{\partial\Omega} \beta |\gamma_\eta(t)|^p d\sigma \right) - \int_{\Omega} \hat{F}(x, \varepsilon \gamma_\eta(t)(x)) dx \\ &< \frac{\varepsilon^p}{p} \left( \Phi(\gamma_\eta(t)) + \int_{\Omega} (\eta - a_1) |\gamma_\eta(t)|^p dx \right) < \frac{\varepsilon^p}{p} \left( \lambda_2 + 2\eta - \text{ess inf}_{x \in \Omega} a_1(x) \right) < 0 \end{aligned}$$

provided  $\eta < \frac{1}{2}(\text{ess inf}_{x \in \Omega} a_1(x) - \lambda_2)$ , because  $\gamma_\eta(t) \in U_C$ . Consequently,

$$\hat{\varphi}|_{\varepsilon \gamma_\eta([-1, 1])} < 0. \quad (3.17)$$

Next, write  $a := \hat{\varphi}_+(u_*)$ . From the proof of Lemma 3.2 it follows  $a < 0$ . We may suppose

$$K(\hat{\varphi}_+) = \{0, u_*\},$$

otherwise the conclusion is straightforward. Hence, no critical value of  $\hat{\varphi}_+$  lies in  $(a, 0)$  while

$$K_a(\hat{\varphi}_+) = \{u_*\}.$$

Due to the second deformation lemma [10, Theorem 5.1.33], there exists a continuous function  $h : [0, 1] \times (\hat{\varphi}_+^0 \setminus \{0\}) \rightarrow \hat{\varphi}_+^0$  satisfying

$$h(0, u) = u, \quad h(1, u) = u_*, \quad \text{and} \quad \hat{\varphi}_+(h(t, u)) \leq \hat{\varphi}_+(u)$$

for all  $(t, u) \in [0, 1] \times (\hat{\varphi}_+^0 \setminus \{0\})$ . Let  $\gamma_+(t) := h(t, \varepsilon \hat{u}_0)^+$ ,  $t \in [0, 1]$ . Then  $\gamma_+(0) = \varepsilon \hat{u}_0$ ,  $\gamma_+(1) = u_*$ , as well as

$$\hat{\varphi}(\gamma_+(t)) = \hat{\varphi}_+(\gamma_+(t)) \leq \hat{\varphi}_+(h(t, \varepsilon \hat{u}_0)) \leq \hat{\varphi}_+(\varepsilon \hat{u}_0) = \hat{\varphi}(\varepsilon \gamma_\eta(1)) < 0; \quad (3.18)$$

cf. (3.17). In a similar way, but with  $\hat{\varphi}_+$  replaced by  $\hat{\varphi}_-$ , we can construct a continuous function  $\gamma_- : [0, 1] \rightarrow X$  such that  $\gamma_-(0) = v_*$ ,  $\gamma_-(1) = -\varepsilon \hat{u}_0$ , and

$$\varphi(\gamma_-(t)) < 0 \quad \forall t \in [0, 1]. \quad (3.19)$$

Concatenating  $\gamma_-$ ,  $\varepsilon \gamma_\eta$ , and  $\gamma_+$  one obtains a path  $\tilde{\gamma} \in \Gamma$  which, in view of (3.17)–(3.19), fulfils (3.14).  $\square$

The next multiplicity result directly stems from Theorems 3.1–3.3.

**Theorem 3.4.** *Let  $(f_1)$ – $(f_4)$  be satisfied and let  $\text{ess inf}_{x \in \Omega} a_1(x) > \lambda_2$ . Then (1.1) admits at least three nontrivial solutions:  $u_0 \in \text{int}(C_+)$ ,  $v_0 \in -\text{int}(C_+)$ , and  $u_1 \in [v_0, u_0] \cap C^1(\overline{\Omega})$  nodal.*

An immediate application of this result produces both constant-sign and nodal solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u - g(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(x) |u|^{p-2} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.20)$$

where  $\lambda > 0$  while  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  denotes a Carathéodory function such that  $g(\cdot, 0) = 0$ . Under Dirichlet boundary conditions, the above equation has been widely investigated; see for instance [21, 4, 19] and the references given there.

**Theorem 3.5.** *Assume that  $\lambda > \lambda_2$ . If, moreover,*

*$(g_1)$  to every  $\rho > 0$  there corresponds  $b_\rho \in L^\infty(\Omega)$  satisfying  $\sup_{|t| \leq \rho} |g(x, t)| \leq b_\rho(x)$  in  $\Omega$ ,*



(g<sub>2</sub>)  $\liminf_{t \rightarrow \pm\infty} \frac{g(x, t)}{|t|^{p-2}t} \geq b_0 > \lambda - \lambda_1$  uniformly with respect to  $x \in \Omega$ ,

(g<sub>3</sub>)  $b_1 \leq \liminf_{t \rightarrow 0} \frac{g(x, t)}{|t|^{p-2}t} \leq \limsup_{t \rightarrow 0} \frac{g(x, t)}{|t|^{p-2}t} \leq b_2 < \lambda - \lambda_1$  uniformly in  $x \in \Omega$ , as well as

(g<sub>4</sub>) for every  $\rho > 0$  there exists  $\mu_\rho > \lambda$  such that  $t \mapsto \mu_\rho |t|^{p-2}t - g(x, t)$  is nondecreasing on  $[-\rho, \rho]$  whatever  $x \in \Omega$ ,

then (3.20) possesses at least three nontrivial solutions:  $u_0 \in \text{int}(C_+)$ ,  $v_0 \in -\text{int}(C_+)$ , and  $u_1 \in [v_0, u_0] \cap C^1(\overline{\Omega})$  nodal.

Conditions (g<sub>2</sub>)–(g<sub>3</sub>) above are much more general than the corresponding ones of [22, Theorem 12] but (g<sub>4</sub>) does not appear in that result. A similar comment holds true for [11, Theorem 3.1], where  $\beta \equiv 0$  and sub-critical behavior for  $t \mapsto g(x, t)$  is taken on. Finally, the  $\beta \equiv 0$  version of Theorem 3.5 and [5, Theorem 4.1] are mutually independent.

## 4 The semilinear case

Suppose  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $f(\cdot, 0) = 0$  and  $f(x, \cdot)$  belongs to  $C^1(\mathbb{R})$  for every  $x \in \Omega$ , while  $f(\cdot, t)$  and  $f'_t(\cdot, t)$  are measurable for all  $t \in \mathbb{R}$ . Let  $F$  be given by (3.1). We will make the following assumptions.

(f<sub>5</sub>)  $|f'_t(x, t)| \leq a_3(1 + |t|^{r-2})$  in  $\Omega \times \mathbb{R}$ , where  $2 \leq r < 2^*$ .

(f<sub>6</sub>)  $f'_t(x, 0) = \lim_{t \rightarrow 0} \frac{f(x, t)}{t}$  uniformly with respect to  $x \in \Omega$ . Moreover, there exists  $m \geq 2$  such that  $\lambda_m \leq f'_t(\cdot, 0) \leq \lambda_{m+1}$ ,  $f'_t(\cdot, 0) \neq \lambda_m$ , and

$$F(x, \xi) \leq \frac{\lambda_{m+1}}{2} \xi^2 \quad \forall (x, \xi) \in \Omega \times \mathbb{R}.$$

(f<sub>7</sub>) If  $m = 2$  then  $\lambda_2 < a_4 \leq f'_t(x, 0)$  uniformly in  $x \in \Omega$ .

It should be noted that (f<sub>5</sub>) implies both (f<sub>1</sub>) and (f<sub>4</sub>) written for  $p = 2$ , while (f<sub>6</sub>) forces (f<sub>3</sub>) with  $p = 2$ . Consider the semi-linear problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where  $\frac{\partial u}{\partial n} := \nabla u \cdot n$ ; see [18, Remark 1.40]. If  $X := H^1(\Omega)$  then the energy functional  $\varphi : X \rightarrow \mathbb{R}$  stemming from (4.1) is

$$\varphi(u) := \frac{1}{2} \left( \|\nabla u\|_2^2 + \int_{\partial\Omega} \beta(x) u(x)^2 d\sigma \right) - \int_{\Omega} F(x, u(x)) dx, \quad u \in X. \quad (4.2)$$

Obviously,  $\varphi \in C^2(X)$  and one has

$$\langle \varphi''(u)(v), w \rangle = \int_{\Omega} \nabla v \cdot \nabla w dx + \int_{\partial\Omega} \beta v w d\sigma - \int_{\Omega} f'_t(x, u) v w dx \quad \forall u, v, w \in X. \quad (4.3)$$

**Lemma 4.1.** *Let (f<sub>5</sub>)–(f<sub>6</sub>) be satisfied. Then  $C_q(\varphi, 0) = \delta_{q, d_m} \mathbb{Z}$  for all  $q \in \mathbb{N}_0$ , where  $d_m := \dim(\bar{H}_m)$ .*

*Proof.* Suppose  $f'_t(\cdot, 0) \neq \lambda_{m+1}$ . By (f<sub>6</sub>), Lemma 2.2 can be applied with  $\theta(x) := f'_t(x, 0)$ . So,  $u = 0$  is a non-degenerate critical point of  $\varphi$  having Morse index  $d_m$ , and the conclusion follows from (2.2). Let now  $f'_t(\cdot, 0) = \lambda_{m+1}$ . Thanks to (f<sub>6</sub>) again, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$F(x, \xi) \geq \frac{1}{2} (f'_t(x, 0) - \varepsilon) |\xi|^2 \quad \text{in } \Omega \times [-\delta, \delta]. \quad (4.4)$$

Since  $\bar{H}_m$  is finite dimensional, we can find  $\rho > 0$  fulfilling

$$u \in \bar{H}_m \cap B_\rho(0) \implies |u(x)| \leq \delta \quad \forall x \in \Omega.$$

Through (4.4) and Lemma 2.2 this entails

$$\begin{aligned} \varphi(u) &\leq \frac{1}{2} \left( \|\nabla u\|_2^2 + \int_{\partial\Omega} \beta u^2 d\sigma - \int_{\Omega} f'_t(x, 0) u^2 dx \right) + \frac{\varepsilon}{2} \|u\|^2 \\ &\leq \frac{1}{2} (-\bar{c} + \varepsilon) \|u\|^2 \leq 0, \quad u \in \bar{H}_m \cap B_\rho(0), \end{aligned}$$

whenever  $\varepsilon < \bar{c}$ . Combining (f<sub>6</sub>) with (2.8) we obtain

$$\varphi(u) \geq \frac{1}{2} \left( \|\nabla u\|_2^2 + \int_{\partial\Omega} \beta u^2 d\sigma - \lambda_{m+1} \|u\|_2^2 \right) \geq 0, \quad u \in \hat{H}_m \cap B_\rho(0).$$

Now, [24, Proposition 2.3] directly yields the conclusion.  $\square$

**Theorem 4.1.** *Under assumptions (f<sub>2</sub>) and (f<sub>5</sub>)–(f<sub>7</sub>), Problem (4.1) admits at least four solutions:  $u_0 \in \text{int}(C_+)$ ,  $v_0 \in -\text{int}(C_+)$ , and  $u_1, v_1 \in \text{int}_{C^1(\bar{\Omega})}([v_0, u_0])$  nodal.*

*Proof.* The existence of  $u_0, v_0, u_1$  comes from Theorem 3.4. Bearing in mind Theorem 3.2 and Lemma 3.2, we may suppose  $u_0, v_0$  extremal constant-sign solutions to (4.1), i.e.,  $u_0 = u_*, v_0 = v_*$ , as well as local minimizers for  $\hat{\varphi}$ . Thus,

$$C_q(\hat{\varphi}, u_0) = C_q(\hat{\varphi}, v_0) = \delta_{q,0}\mathbb{Z} \quad \forall q \in \mathbb{N}_0; \quad (4.5)$$

see [18, Example 6.45]. Let us next verify that  $u_1 \in \text{int}_{C^1(\overline{\Omega})}([v_0, u_0])$ . Put

$$\rho := \max\{\|u_0\|_\infty, \|v_0\|_\infty\}.$$

If  $\mu_\rho$  is as in (f<sub>4</sub>) then

$$-\Delta(u_0 - u_1) + \mu_\rho(u_0 - u_1) = [f(x, u_0) + \mu_\rho u_0] - [f(x, u_1) + \mu_\rho u_1] \geq 0$$

because  $u_1 \leq u_0$ . So, by [25, Theorem 5],  $u_0 - u_1 \in \text{int}(C_+)$ . Likewise,  $u_1 - v_0 \in \text{int}(C_+)$ , and the assertion follows.

The proof of Theorem 3.3 ensures that  $u_1$  is a Mountain Pass type critical point for  $\hat{\varphi}$ . Thanks to (4.3), Corollary 6.102 in [18] gives

$$C_q(\hat{\varphi}, u_1) = \delta_{q,1}\mathbb{Z}, \quad q \in \mathbb{N}_0. \quad (4.6)$$

From  $\hat{\varphi}|_{[v_0, u_0]} = \varphi|_{[v_0, u_0]}$  and Lemma 4.1 we infer

$$C_q(\hat{\varphi}, 0) = \delta_{q, d_m}\mathbb{Z} \quad \forall q \in \mathbb{N}_0, \quad (4.7)$$

while the coercivity of  $\hat{\varphi}$  entails (cf. [18, Proposition 6.64])

$$C_q(\hat{\varphi}, \infty) = \delta_{q,0}\mathbb{Z}, \quad q \in \mathbb{N}_0. \quad (4.8)$$

Now, if  $K(\hat{\varphi}) = \{0, u_0, v_0, u_1\}$  then the Morse relation (2.1) written for  $t = -1$  and (4.5)–(4.8) would imply

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0,$$

which is impossible. Thus, there exists a further point  $v_1 \in K(\hat{\varphi}) \setminus \{0, u_0, v_0, u_1\}$ . Lemma 3.1, combined with (3.9), shows that  $v_1$  turns out to be a nodal solution of (4.1) that lies in  $[v_0, u_0]$ . Standard regularity arguments provide  $v_1 \in C^1(\overline{\Omega})$ . Finally, reasoning as before one achieves  $v_1 \in \text{int}_{C^1(\overline{\Omega})}([v_0, u_0])$ .  $\square$

An immediate application of this result produces both constant-sign and nodal solutions to the problem

$$\begin{cases} -\Delta u = \lambda u - g(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.9)$$

where  $\lambda > 0$  while  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  denotes a function such that  $g(\cdot, 0) = 0$  and  $g(x, \cdot)$  belongs to  $C^1(\mathbb{R})$  for every  $x \in \Omega$ , while  $g(\cdot, t)$  and  $g'_t(\cdot, t)$  are measurable for all  $t \in \mathbb{R}$ .

**Theorem 4.2.** *Let  $\lambda \in (\lambda_m, \lambda_{m+1}]$  for some  $m \geq 2$  and let  $(g_2)$  of Theorem 3.5 be satisfied with  $p = 2$ . If, moreover,*

$$(g_5) \quad |g'_t(x, t)| \leq b_3(1 + |t|^{r-2}) \text{ in } \Omega \times \mathbb{R}, \text{ where } 2 \leq r < 2^*,$$

$$(g_6) \quad g'_t(x, 0) = \lim_{t \rightarrow 0} \frac{g(x, t)}{t} = 0 \text{ uniformly with respect to } x \in \Omega, \text{ and}$$

$$(g_7) \quad \int_0^\xi g(x, t) dt \geq \frac{\lambda - \lambda_{m+1}}{2} \xi^2 \text{ for all } (x, \xi) \in \Omega \times \mathbb{R},$$

*then the same conclusion of Theorem 4.1 holds true concerning (4.9).*

The sign condition  $tg(x, t) \geq 0$ ,  $(x, t) \in \Omega \times \mathbb{R}$ , clearly forces  $(g_8)$ . So, Theorem 4.2 basically extends [22, Theorem 14]. For  $\beta \equiv 0$ , cf. also [8, Theorem 3.7], [11, Section 4], and the references given there.

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